Money with Partially Directed Search

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Abstract

Partially directed search replaces the total randomness of monetary search models. Agents choose whether to stay in their production location or visit other locations. Each visiting agent randomly chooses one shop among many in each location. As in random matching models, a commodity or fiat object can endogenously become money, but the details are richer and conform better with evidence: any commodity can be money; the "best" commodity is the most likely money; fiat money can totally crowd out commodity money in an asymmetric environment; going shopping is more likely than door-to-door sales. The model nests Walrasian equilibria.

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1 Introduction

The random matching model has become the workhorse of decentralized monetary economics. While many of the technical restrictions of Kiyotaki and Wright (1989, henceforth KW) have been relaxed in the subsequent literature, one major restriction inherently remains in all the random matching models. This is the random matching property itself. Agents cannot choose whom to meet or where to be. They behave as "particles colliding in space."[1] Random matching is very tractable but it is unrealistic. Modern real-life trade is almost as far from the total chaos of random matching as it is far from the perfect coordination of the Walrasian framework. This may be one major reason that the results of random matching models are still treated with some suspicion by many economists.

One may wonder what is the nature of monetary equilibria when there is *partially directed search*, as in real life, rather than total randomness. *Partially directed search* has a specific meaning here. It means that the search process has two stages—a directed stage and a random stage. In the directed stage agents deliberately and actively get access to potential sellers of the good they desire. Then, when they face many ex-ante identical sellers, comes the second stage: they randomly choose which particular seller to visit. Such a search process is a reasonable characterization of many trade activities. For example, someone who needs a taxi cab can look in the Yellow Pages under "Taxi" and then randomly choose one of those listed there. Going to the marketplace or the mall, or typing "shoes" in Yahoo! are other examples of the first step in this search process. The second, random stage keeps money essential.

The comparison between partially directed search and random matching is facilitated by using a model which is otherwise identical to KW. Agents have only one more piece of information, and choose only one more strategy, beyond that which they have and choose in KW. They know where all the fixed production locations are, and choose in which *specific* production location to be. In equilibrium, it is endogenously and simultaneously determined who offer goods in their production locations (i.e., open shops), and which objects become money.

It turns out that the main results of the standard random matching models are preserved: a commodity or fiat money can arise endogenously in equilibrium. However, the current model has three sets of useful results, in which it improves on random matching models. One set of results has to do with properties of the monetary equilibria. The relations between commodity money and fiat money, and between various potential commodity moneys, conform better with evidence from monetary history. Another set of results yields predictions about geographical trade patterns and relations between choices of shops and choices of money, which cannot even be discussed in a random matching model. Finally, there is much more order in equilibrium. The model therefore nests Walrasian equilibria, which realize if all agents happen to meet in the same place, many of its results easily generalize to any number of types of goods and agents, and

[1] Burdett, Coles, Kiyotaki and Wright (1995).

most of its results do not depend on the particular pattern of specialization in production.

There are other models in which trade is more organized than in random matching models. Burdett, Coles, Kiyotaki and Wright (1995) let agents choose between standing still and wandering around. The latter (costly) option increases the probability of randomly meeting other agents. Here the incentive not to stand still is the ability to go directly to someone else's production location. Their equilibrium in which only fiat money is used is achieved by crowding out direct barter rather than commodity money (commodity money is not analyzed in that model). Trading posts have been widely used in the literature, where a trading post (i, j) is a location where *deterministically* only objects i and j are exchanged for each other. The exchange process within a post can be centralized (Shapley and Shubik [1977]), managed by a specialized intermediary (e.g., Howitt and Clower [2000]), or conducted by random matching (e.g., Iwai [1988], Kultti [1994], Kranton [1996]). In these models it is not clear how the posts come into existence, who enforces their strict rules, and how agents know without costly advertising where each trading post is[2]. In contrast, the shops used in the current paper are the natural pre-existing production locations, and any object may be traded at any location. In fact, for every pair (i, k), where i is a good and k is a production location, there is an equilibrium in this paper in which good i is traded in location k. Banerjee and Maskin (1996) discuss directed search in an almost frictionless model, which is very different from random matching models. Corbae, Temzelides and Wright (2003) analyze *cooperative* endogenous matching. They also focus on comparison with random matching models, and reach results which are different from both random matching and the non-cooperative partially directed search that is presented here.

To facilitate the presentation of the numerous results the paper proceeds gradually. In Section 2 the basic economy is presented. Unlike KW, it has complete symmetry across goods. This model is analyzed in Sections 3 (autarky), 4 (Walrasian equilibria) and 5 (commodity money). Fiat money is added in Section 6. Section 7 gets closest to KW and reality by assuming that goods have different intrinsic properties. Section 8 concludes. All the proofs are in the Appendix.

2 The Model

To facilitate the comparison between a partially directed search model and a random matching model, for both commodity and fiat money, the model and notation are as similar as possible to Kiyotaki and Wright (1989). The description begins with those features of the model that are identical to KW, and is followed by the new features.

[2] In Matsui and Shimizu (2005) a trading post equilibrium endogenously emerges in a less deterministic environment, in which any real good can be sold at any post, but all trade must involve fiat money.

2.1 Standard Features

This is a discrete time infinite horizon economy with indivisible goods 1, 2, and 3. There is a continuum of infinitely lived agents with unit mass. There are equal proportions of agents of types 1, 2, and 3. Agents specialize in both consumption and production. Type n agents consume only good n, $n = 1, 2, 3$. The specialization in production is general enough to include both KW's Model A and Model B. Type i agents produce only another good j, type j agents produce only another good k, and type k agents produce only good i. The discussion of equilibria below is in terms of types i, j, k. There are six interesting assignments of each element of $\{1, 2, 3\}$ to each element of $\{i, j, k\}$. The first three describe KW's Model A and are collected in the set $\{(1, 2, 3), (2, 3, 1), (3, 1, 2)\}$;[3] the other three describe KW's Model B and are collected in the set $\{(1, 3, 2), (2, 1, 3), (3, 2, 1)\}$.

Agents derive utility U from consuming one unit of their desired good and disutility D from producing one unit of their production good, with $u \equiv U - D > 0$.[4] Production must be preceded by consumption so with an initial endowment of one good per agent no one ever holds more than one unit of any good. Let V_{ij} be the value function of a type i agent holding good j. The proportion of such agents among all type i agents is p_{ij}. The discount factor is $\beta \in (0, 1)$.

2.2 The Geographical Environment

To introduce partially directed search in a tractable way, the following geographical environment is assumed. Every agent owns a structure that has several functions. First, it is his home. This is where he must be at the beginning and the end of every period, where the period may be thought of as a day. Second, it is his factory. This is where he produces his production good at the end of the day. Third, it is his storage place. He can store there his production good or any other good. Fourth, it *may* function as a shop. This is the only place in which he can offer visitors any object for trade. All the factories of type n agents are located in the same place, called City n.[5] This is the only place where their common production good can be produced. All agents know where these cities are located and which goods are produced in each city. This piece of information is the critical difference between this model and KW.

Agents not only know more than in KW, but they also have the ability to act on that additional information. Every morning each agent chooses which

[3] In some of the analysis below it makes a difference which good is denoted good i in each production structure (e.g., this will be the candidate for commodity money).

[4] In KW the utility parameters vary across agents. This assumption did not affect the results and was abandoned in subsequent literature. It also does not make much difference in this paper. For an investigation of commodity money, obviously intrinsic differences across goods are more interesting than differences in agents' utility parameters. For an analysis with type–specific utility parameters, see Goldberg (2002).

[5] In the literature locations are referred to as "islands" when the spatial separation *restricts* information. In contrast, here the existence of fixed natural production locations *enhances* information. The word "city" should also remind that there are infinitely many factories in each location.

city to be in for the day. He can stay home and open a shop (then he is called a *stayer*). Alternatively he can go to another particular city (then he is called a *searcher*). The travel cost for any searcher, from his factory to someone else's factory or back, is represented by the weight of the good he carries. Good n weighs c_n, where $0 \leq c_1 \leq c_2 \leq c_3$.[6] Stayers do not incur any such transport costs, nor do they have any storage costs. By evening all searchers must go back to their homes, regardless of whether they traded or not. Successful searchers can consume on the spot, so they do not incur transport costs on the way back.

One parable is of a primitive economy in which all goods are natural resources, and therefore must be produced in fixed specific locations: fish are produced in the fishermen's village near the lake, vegetables are produced in the farmers' village in the field, and meat is produced in the hunters' village in the plain. It is reasonable to assume that such an economy eventually converges to a steady state where this information is common knowledge and costless[7]. Another parable is the Yellow Pages, where all producers of a good are merely listed together under the same category. In both cases, a searcher has free information that enables him to direct his search toward that group of producers.

2.3 Matching

All type n searchers who visit some City l arrive there together. They face infinitely many factories around them. They do not know which factories are open as shops and what good is actually offered in each shop[8]. Since all factories are ex-ante identical every searcher chooses randomly which factory to visit. This random choice of a factory represents the second stage in the partially directed search process described above. In the model it serves to preclude bilateral credit, so trade must be quid pro quo as in KW.

In order to maintain the bilateral matching of KW as much as possible, it is assumed that the short side of the market is fully matched. That is, if there are fewer searchers than factories, each factory is visited by one searcher at most[9]. If there are fewer factories than searchers, each factory is visited by at least one searcher. Two searchers of the same type never reach the same factory[10]. This rule in an economy of three types implies that each factory, whether it is open as a shop or not, is visited by zero, one, or two searchers.

[6] With the interpretation that KW's agents move all the time like particles in space, their storage costs can be thought of as transport costs. In KW the storage costs also depend on the agents' types, but this is not important for the results in either KW or here.

[7] As a modern illustration, energy companies know that a lot of oil is produced in the Middle East, although they did not pay anyone to get this information, and the oil producers currently do not advertise their locations.

[8] Giving them such information will make the model more realistic, but will also take it further away from random matching models—and this is not useful for the purposes of this paper.

[9] A parable of this matching rule, which has been used elsewhere, is that of taxi cabs that wait in line in front of an airport terminal. Each passenger that exits the terminal goes to the cab which is currently the first in line.

[10] Since the proportions of types are identical, there are always enough factories for this matching rule.

When a shop is visited by two searchers, the stayer can trade with one of the searchers or both[11]. If he is indifferent between them he conducts a lottery. A searcher must return home after visiting one factory, regardless of the outcome of that match, and even if the factory he chooses is not open as a shop at all. He can try again the next day. In order to clearly distinguish between random matching and partially directed search, it is assumed that searchers can trade with each other only when a stayer is involved in the exchange. They cannot trade in his shop if he is not there, they cannot trade behind his back, and they cannot meet outside shops.

2.4 Strategies and Equilibrium

The strategies are more complicated than in random matching models because an agent chooses both a trading strategy (whether or not to exchange goods) and a specific location strategy within a partially directed search setting. The lack of a social arrangement that creates a marketplace for all goods in a certain location, requires agents to guess where they can find the goods that they want, and how they can pay for these goods[12]. There are three possible location strategies. The value function of a type i agent who holds good n at the end of the period is

$$V_{in} = \beta \max[\text{value of staying to City } i, \text{ value of going to City } j, \\ \text{value of going to City } k].$$

The equilibrium concept is otherwise the same as in KW.

Definition 1 *A Nash equilibrium is a set of pure and stationary trading strategies, pure and stationary location strategies, and a stationary distribution of goods, such that: (i) each agent chooses strategies to maximize expected utility given others' strategies and the distribution; (ii) given the strategies, the result is that distribution.*

Following KW, it is assumed that agents who are indifferent between trading and not trading do not trade. In the same spirit, it is assumed that agents who are indifferent between staying home and searching stay home[13].

In KW storage costs are unavoidable. This may result in negative value functions, and may induce an agent to forgo consumption in order to avoid the possibility of subsequently holding his very costly-to-store production good forever. Their Assumption A and Lemma 1 address these concerns. Lemma 1 below shows that these problems do not exist with the transport costs assumed here, because they can be avoided by staying home.

[11] In Goldberg (2002) I examine the case where trilateral meetings are impossible.

[12] Going back to the parable of a primitive economy, a farmer who wants to consume fish can go to the fishermen or wait for them to come to him. He can also speculate that they might have used the fish to buy meat from the hunters, in which case he may visit the hunters.

[13] These assumptions can be justified by an arbitrarily small cost of not being idle.

Lemma 1 *(i) Value functions are non-negative; (ii) agents always accept their consumption goods and consume them immediately.*

Finally, if an agent holds a good that turns out to function as money, then he is called a *buyer*; otherwise, he is called a *seller*.

3 Autarky

There is a possibility that location strategies imply no matching at all, and thus an autarkic equilibrium. Consider the case where no agent chooses to be a searcher. If this is an equilibrium agents will never meet each other and will always hold their production goods. However, construction of the equilibrium requires a complete specification of strategies. Suppose then that agents accept only their consumption goods, and that both this trading strategy and the location strategy above apply no matter what the agents hold.

If a type i agent believes that everyone else follows these strategies, his optimal strategy is defined by the following equations

$$V_{ij} = \beta \max[V_{ij},\ -c_j + \max(V_{ij} - c_j,\ V_{ik} - c_k),\ -c_j + V_{ij} - c_j], \tag{1}$$

$$V_{ik} = \beta \max[V_{ik},\ -c_k + V_{ik} - c_k,\ -c_k + u + V_{ij}]. \tag{2}$$

In (1), an agent who holds good j has three options. If he stays home nothing happens because nobody visits him. If he goes to City j he incurs a transport cost on his way there. Since he is the only searcher around and type j agents accept his good j, he can get good k for sure, if he wants to. Whether he decides to trade or not, he incurs another transport cost when he goes back home. The last option is to go to City k, where type k agents live. On his way there he incurs a transport cost. Since type k agents accept only good k he cannot trade and he takes good j back home. In (2), he cannot trade in City j because everyone there already has the good k he brings. If he goes to City k, he is the only searcher there and the locals want his good k. Therefore, he gets good i for sure, consumes it on the spot, goes back home, and produces another unit of good j. The next step is to guess that the agent follows the same strategies as everyone else, and then try to verify the guess. It turns out that it is optimal to stay home iff the transport costs are too high.

Proposition 1 *Autarky with no searchers is an equilibrium iff $u \leq c_1$.*

The same result appears in Burdett, Coles, Kiyotaki and Wright (1995), for the same reason. It is important to note that this equilibrium's trading strategies in themselves do not imply autarky, as demonstrated in the Walrasian equilibria of the next section[14]. The only problem is the location strategies.

[14] If it is assumed that agents who are indifferent do trade, then there is a similar autarky equilibrium in which agents agree to accept all goods.

The opposite location strategies do not lead to equilibrium.

Proposition 2 *Autarky with no stayers is not an equilibrium.*

If one believes that everyone else are searchers, it is optimal to stay home. This is the only chance of meeting anyone, and it saves on transport costs.

In fact, a stronger statement can be made.

Proposition 3 *Autarky with any searchers is not an equilibrium.*

In equilibrium agents' expectations regarding others' behavior are correct. If an agent expects that others' behavior imply autarky, there is no point in searching.

4 Walrasian Equilibria

Suppose that all type i agents and all type k agents take their production goods to City j and all type j agents stay there. Suppose also that they all accept only their consumption goods. Then in each shop there are a type j stayer, a type i searcher, and a type k searcher. There is no need for money: each type i agent gives his production good to the agent who consumes it. The matching frictions are completely overcome since the agents happen to coordinate on the "marketplace." Of course, the same can happen in any other city.

Proposition 4 *A Walrasian equilibrium centered in City j exists iff $u > \max(c_j, c_i)$, where i is not type j's production good.*

As in Proposition 1, the transport costs must be low enough to enable trade. In particular, both of the goods that are not produced in City j (good j is one of them) must be light enough to be transported to City j. None of these equilibria Pareto dominates the others because each type is better off in the equilibrium in which his city is the marketplace.

5 Commodity Money

Suppose that type k agents take their production good (good i) to City j and trade it for good k. Suppose further that type j agents then take that good i to City i and trade it for good j. If this happens in equilibrium then good i functions as *commodity money* and type j agents function as *intermediaries*, as in KW's *fundamental equilibrium.* Moreover, this equilibrium can be defined as a *shopping equilibrium*: both type j agents and type k agents trade by taking the commodity money to shops that sell their consumption goods. In other words, all searchers are buyers and all stayers are sellers. The opposite case is when type j agents go to City k, take good i from there, and are then visited in City j by type i agents. Good i is still commodity money and type j agents are still intermediaries, but now the agents who trade good i for another good do so while in their homes. Thus, this is defined as a *door-to-door equilibrium.* There is only one other possible trading pattern in which good i is the unique money, and this is when type j agents, the intermediaries, are the only ones who leave their homes: they go first to City k and, after resting overnight at home, they then go to City i. This is defined as a *one-searcher equilibrium.* The opposite of this equilibrium is when only type j agents stay home, but this is the same Walrasian equilibrium analyzed above[15].

Figure 1 shows an illustration of all these trade patterns. The case of more than one commodity money is discussed later. Table 1 explains how any assignment of the numbers 1, 2, 3, to each of the letters i, j, k, corresponds to a particular commodity money equilibrium in either Model A or Model B. This is useful for understanding the following three propositions.

Proposition 5 ***(i)*** *When $c_i \leq c_j$ the shopping equilibrium with good i as money exists iff $u > 3c_i$;* ***(ii)*** *When $c_i > c_j$ the shopping equilibrium with good i as money exists iff $u > \max\left[3c_i,\ \frac{(2+5\beta)c_i - 4(1+\beta)c_j}{2-\beta}\right]$.*

To understand the main condition in the proposition, consider the case where good 1 is the candidate money in Model A. In the steady state of the shopping equilibrium, half of type 2 agents are in City 2, visited by all type 3 agents, and half of type 2 agents are in City 1, visiting all type 1 agents. Therefore, in City 2, where good 3 is produced, the demand for good 3 is twice the supply. A type 3 agent that visits City 2 incurs a cost c_1 on his way there. With probability .5 he buys and consumes. With probability .5 he visits a factory whose owner is in City 1. Then he goes back home with the same good he brought. Thus, a visit to City 2 yields an expected payoff of $-c_1 + .5u - .5c_1$, so it is optimal to visit City 2 iff $u > 3c_1$. In contrast, type 2's visit to City 1 is sure to succeed. The demand there for good 1 is half the supply, so the visiting agent gets good 1 for sure. That travel is optimal iff $u > c_1$.

The other equilibria are not as likely to exist.

Proposition 6 ***(i)*** *When $c_j \geq c_k$ the door-to-door equilibrium with good i as money does not exist;* ***(ii)*** *when $c_j < c_k$ the door-to-door equilibrium with good i as money exists iff $\max(3c_j,\ \frac{c_i+c_k}{\beta}) < u \leq \frac{2(1-\beta)c_k-(2+3\beta)c_j}{\beta}$;* ***(iii)*** *if $\beta \geq .5$ no door-to-door equilibrium exists;* ***(iv)*** *if agents who are indifferent between accepting a good and rejecting it agree to trade, then no door-to-door equilibrium exists.*

Proposition 7 ***(i)*** *When $c_i \geq c_k$ the one-searcher equilibrium with good i as money does not exist;* ***(ii)*** *when $c_i < c_k$ the one-searcher equilibrium with good i as money exists iff $\frac{(1+\beta)c_i+c_k}{\beta} < u \leq 2c_k$;* ***(iii)*** *if agents who are indifferent between accepting a good and rejecting it agree to trade, then an additional condition is $u \leq 2c_j$.*

[15] In Goldberg (2002), where meetings must be bilateral, this is a commodity money equilibrium called a one-stayer equilibrium. It is very similar in properties and robustness to the shopping equilibrium because in both equilibria the money's producers travel and are thus not directly accessible to the money's consumers.

These two equilibria types are much more fragile than the shopping equilibrium. The fragility of the door-to-door equilibrium is significant as in any reasonably calibrated model $\beta > .5$. Continuing the example of the shopping equilibrium above (with good 1 as money in Model A), note that the producers of the candidate money (type 3 agents) stay home in the door-to-door and one-searcher equilibria. This is the critical difference. As opposed to the shopping equilibrium, now they are accessible to type 1 agents (see Figure 2). A type 1 agent is supposed to get good 1 from a type 2 agent in these equilibria, but if a type 2 agent deviates and offers him good 3, it may be optimal for him to accept it. The reason is that he can take it to City 3, where he will be on the short side of the market and matched for sure (all type 3 agents are there, offering good 1, and the only other visitors are half of type 2 agents). If he waits for a type 2 agent to obtain good 1 for him then he is on the long side of the market (half of type 2 agents hold good 1, and all type 1 agents want to match with them). Given that type 1 agents accept good 3, type 2 agents do not need to get good 1 from type 3 agents in order to trade with type 1 agents. This means that good 1 is not money.

The impact of these matching frictions is clearly seen when transport costs are symmetric. Then both of these equilibria do not exist. With asymmetric transport costs, the model has some degrees of freedom so these equilibria can exist. Also note that if a type 1 agent accepts good 3 from a type 2 agent (in order to take it to City 3), then good 3 becomes money instead of good 1. In the competition between goods on the role of money, there is a presumption against the one that is offered in shops by its producers.

Even without these fragile equilibria, Proposition 5 alone already has a very important implication.

Theorem 1 *Any good can become commodity money.*

In spite of this indeterminacy, there is a prediction regarding which equilibrium is more likely to exist than others.

Proposition 8 *(i) The set of parameter values under which the shopping equilibrium with good 1 as money exists is a superset of the sets of parameter values under which all the other commodity money equilibria exist; (ii) only the shopping equilibrium has a strict monotonicity in the following sense: if it exists with good x as money (in Model A), then it also exists with good y as money for all y that satisfy $c_y < c_x$.*

Figure 3 demonstrates both parts of the proposition for $\beta = 1$ and $c_1 = 0$ in Model A. Part (ii) follows from the fact that the shopping equilibrium is the only one in which only the money is transported (see Figure 1).

The model thus gives a plausible explanation for three facts about the history of commodity money:

Fact 1. Trade was typically conducted by "going shopping."[16]

[16] Door-to-door sales were invented as an organized form of trade in the U.S. only in 1906 (http://www.salesdoor2door.com).

Fact 2. Usually the commodity that was the most suitable to function as money due to its physical properties indeed functioned as money[17].

Fact 3. Notwithstanding Fact 2, important deviations did occur. Almost any pre-industrial commodity functioned as commodity money at some place at some time[18].

Welfare results are ambiguous.

Proposition 9 *Among the commodity money equilibria no type of equilibrium is unambiguously the best or the worst.*

The shopping equilibrium is efficient in the sense that all successful buyers carry nothing home (they consume on the spot). The one-searcher equilibrium is efficient in the sense that all searchers are successful: nobody incurs a transport cost in vain because of matching frictions. Combining these considerations with the ranking $c_1 < c_2 < c_3$ leads to ambiguous welfare ranking[19].

The matching frictions of the commodity money equilibria make them dominated in every possible way by the Walrasian equilibria if transport costs are the same.

Proposition 10 *If $c_i = c_j = c_k$: (i) if any shopping equilibrium exists, then all the Walrasian equilibria exist too; (ii) the shopping equilibrium with good i as money is Pareto dominated by the Walrasian equilibrium in which all trade is in City i; (iii) aggregate welfare is lower in any shopping equilibrium than in any Walrasian equilibrium.*

However, asymmetric transport costs can make a difference. Walrasian equilibria then no longer dominate all commodity money equilibria.

Proposition 11 *If $c_1 < c_2 < c_3$: (i) The set of parameter values under which the shopping equilibrium with good 1 as money exists can be a superset of the sets of parameter values under which all the Walrasian equilibria exist; (ii) this shopping equilibrium is not necessarily Pareto dominated by any Walrasian equilibrium; (iii) aggregate welfare can be higher in this shopping equilibrium than in one of the Walrasian equilibria.*

The shopping equilibrium has the unique advantage that only one good—potentially the lightest one—is ever carried in it, while any Walrasian equilibrium requires transporting two types of goods[20]. This can explain why money has existed even in very small societies that probably could have organized Walrasian exchange[21]. Interestingly, this preference for money is not necessarily a

[17] See the detailed survey by Einzig (1966).

[18] Again, see Einzig (1966).

[19] E.g., among equilibria with good 1 as money in Model A, the shopping equilibrium is better than the one-searcher equilibrium if $2c_1 < c_3$, even though it has more matching frictions (see the discussion after Proposition 5).

[20] E.g., if good 2 is much heavier than good 1, then the shopping equilibrium with good 1 as money dominates the best Walrasian equilibrium (in which the two lightest goods—goods 1 and 2—are transported).

[21] Einzig (1966) shows that independent monetary systems have existed on a very large number of tiny Pacific islands.

bad one. It turns out that the asymmetry between goods not only explains why some goods are more likely than others to be money, but also why monetary exchange can be more likely than, and better than, Walrasian exchange[22].

The propositions above regarding autarky, Walrasian equilibria, and commodity money, imply the following.

Theorem 2 *For some parameter values the shopping equilibrium with good 1 as money is the unique equilibrium.*

There is a striking similarity in the role of good 2 in the door-to-door equilibria and in KW. In both cases good 2 is the only good which is not money in Model A (Proposition 6(i); KW's Theorem 1) but the most likely money in Model B (Proposition 6(i); KW's Theorem 2). The source of the similarity is seen in Figure 1: the door-to-door equilibrium is the *only* non-Walrasian equilibrium in which the consumer of the commodity money is a searcher (and thus incurs transport costs). Moreover, he can meet agents of any other type because there are stayers of both types 2 and 3. This is as close as the current model can get to KW, where the costs are storage costs—always incurred by everyone—and agents can be matched with agents of any type. Another manifestation of this relation is the fact that all door-to-door equilibria and all KW's equilibria do not exist with symmetric transport costs.

In KW most equilibria have two moneys but this cannot happen here.

Proposition 12 *There is no equilibrium with two commodity moneys.*

For example, for both goods 1 and 3 to be money, there must be meetings between type 1 agents and type 2 agents to make good 3 money, and meetings between type 2 agents and type 3 agents to make good 1 money, when all hold their production goods. With pure, stationary location strategies of type 2 agents all these meetings are in City 2, and this is the Walrasian, money-less equilibrium described above[23].

This model allows any object to be traded at any location (unlike trading posts models). This is not only permissible; in fact, almost anything can happen in any city in equilibrium.

Proposition 13 *(i) For every pair (n, h), where $n \in \{1, 2, 3\}$ is a good and $h \in \{1, 2, 3\}$ is a city, there is a commodity money equilibrium in which good n is traded in City h; (ii) for almost every triple (n, l, h), where $l \in \{1, 2, 3\}$ is another good, there is an equilibrium in which good n is traded bilaterally for good l in City h; (iii) in every city there is an equilibrium with a trilateral exchange of all goods; (iv) for every city there is at least one equilibrium in which nothing is traded there.*

[22] Obviously the assumption of consumption on the spot is critical for this proposition.

[23] The phenomenon of multiple commodity moneys was observed in many societies. It can be easily generated in this model not only by making strategies mixed or non-stationary, but also by increasing the number of types of goods and agents. An example of two commodity moneys with four types is available by request. In general, with N types, it is probably the case that no equilibirum has $N-1$ moneys.

In part (ii), "almost" means eight out of nine possible exchanges in Model A and seven out of nine in Model B. Of course, not all these equilibria coexist for given parameter values.

6 Fiat Money

Here fiat money is added to the economy. It is indivisible and has a transport cost $c_0 \geq 0$. A fraction $m \in (0,1)$ of agents of each type are endowed with one unit of fiat money instead of a real good. The analysis is restricted to equilibria in which fiat money totally crowds out commodity money.

Definition 2 *A fiat equilibrium is an equilibrium in which: (i) fiat money is used in all transactions; (ii) agents never accept real goods that they do not consume.*

Consider a *shopping* fiat equilibrium and a type i agent who expects the following: all agents who hold their production goods open shops and agree to accept only their consumption goods (first priority) or fiat money; when they get fiat money they go shopping in the city that produces their consumption goods. His equations are

$$\begin{aligned} V_{i0} &= \beta \max\{V_{i0}, \\ &\quad -c_0 + m(V_{i0} - c_0) + (1-m)\max(V_{i0} - c_0,\ V_{ik} - c_k), \\ &\quad -c_0 + m(V_{i0} - c_0) + (1-m)(\ u + V_{ij})\}, \end{aligned} \tag{3}$$

$$\begin{aligned} V_{ij} &= \beta \max[m \max(V_{ij},\ V_{i0}) + (1-m)V_{ij}, \\ &\quad -c_j + (1-m)\max(V_{ij} - c_j,\ V_{ik} - c_k) + mV_{ij},\ \ -2c_j + V_{ij}], \end{aligned} \tag{4}$$

$$V_{ik} = \beta \max[V_{ik},\ \ -2c_k + V_{ik},\ \ -c_k + m(V_{ik} - c_k) + (1-m)(u + V_{ij})]. \tag{5}$$

In this equilibrium, among each type, the m buyers are searchers and the $1-m$ sellers are stayers. A searcher finds an open shop with probability $1-m$. Only searchers of one type are in each city, so a factory is visited with probability m. In (3), a money holder who stays home is visited only by money holders, so he does not trade. If he goes to City j he finds a closed shop with probability m and an open shop with probability $1-m$. He can get good k if he wants to. If $I = 0$ the stayer faces two agents of different types who hold the same object (fiat money), so there is a lottery. If the type i agent goes to City k and finds an open shop he can consume. In (4), an agent staying home with his production good may be visited by a buyer who holds fiat money. If he goes to City j he will be offered good k. If he tries to buy his consumption good i with good j in City k, it is rejected. In (5), staying home with good k or taking it to City k does not lead to trade because all type j agents reject their own production good. Going to City k leads to trade of good k for good i and consumption.

Proposition 14 *(i) A shopping fiat equilibrium exists for some parameter values; (ii) a necessary condition for existence is $c_0 \leq c_1$.*

Fiat money must be the lightest object for exclusive circulation. To see why, consider a type i stayer. He is supposed to trade only with a searcher who holds good 0, which he will later take to City k. If $c_0 > c_k$ he prefers to do the same with good k instead. Knowing that, a type j agent will visit him with his production good (good k) instead of going through the trouble of obtaining fiat money first. Therefore, it must be the case that for any type i agent $c_0 \leq c_k$, or $c_0 \leq c_1$.

The existence of this equilibrium is worth emphasizing.

Theorem 3 *Fiat money can entirely eliminate commodity money in an asymmetric environment.*

The real world is asymmetric. Some goods, such as gold, are better than others as money. Such commodity moneys dominate monetary history but were entirely crowded out by fiat money. This model can generate this outcome, while random matching models cannot (Aiyagari and Wallace [1992]). Kultti (1994) and Burdett, Coles, Kiyotaki and Wright (1995) do get equilibria where only fiat money circulates, but only by crowding out *direct* barter in the *symmetric* environment of Kiyotaki and Wright (1993).

The excessive traveling in this equilibrium has negative implications, regardless of how light the fiat money is.

Proposition 15 *(i) The shopping fiat equilibrium does not Pareto dominate any non-autarkic equilibrium; (ii) the shopping fiat equilibrium has lower aggregate welfare than any other non-autarkic equilibrium that coexists with it.*

The Pareto result stems from the fact that in any commodity money equilibrium and any Walrasian equilibrium there is at least one type of agents that never moves, while in the shopping fiat equilibrium all agents move. This result is robust to the particular assumptions on the matching process. However, the aggregate welfare result does depend critically on the assumed matching process. If agents could direct their search inside a city towards the open shops, and $m = .5$, there would have been no matching frictions in the fiat equilibrium and then it could have been better than all other equilibria. Even with the original assumption, fiat money can still be helpful if the parameter values are such that the only alternative is autarky.

In a door-to-door fiat equilibrium all agents take their production goods to their consumers; they accept only fiat money and their consumption goods; when they get fiat money they take it home, and there they accept only their consumption goods.

Proposition 16 *The door-to-door fiat equilibrium does not exist.*

It is optimal for everyone to deviate from such an equilibrium. For example, a type j agent who deviates by staying home with good k, his production good, still gets good j from the type i searcher visiting him. That searcher agrees to trade because he gets good k, which will be accepted by the type k searcher who will visit him next period.

Another result worth emphasizing follows from the propositions above.

Theorem 4 *Trade is much more likely to occur by "shopping" than by door-to-door sales, for both commodity money and fiat money.*

This dominance conforms well with historical evidence[24].

The model is also easier to generalize than random matching models.

Proposition 17 *The results regarding autarky, Walrasian equilibria and fiat equilibria hold for the general model with more than three types of goods and agents.*

The intuition regarding the non-existence of the door-to-door fiat equilibrium is a bit different from above. Consider an extension of Model A with four types of goods and agents, with type i producing good $i+1$ (mod 4). Suppose that type 4 deviates and stays home with his production good (good 1). He is visited by a type 3 agent who expects to be paid with fiat money. Type 3 will have to resell good 1 if he accepts it. He can do that by selling it in City 1 for fiat money. Alternatively, he can reject good 1 and try again the next day to sell good 4 in City 4. The probability of selling good 1 in City 1 equals the probability of selling good 4 in City 4. Therefore he is better off accepting good 1, the lighter good. Then type 4 indeed stays home with good 1 and the equilibrium breaks down. A type whose production good is lighter than his consumption good deviates.

This means that there is no need to resort to symmetry across goods or to their perishability to derive a fiat equilibrium with many types.

7 Conclusion

Total randomness in matching is not necessary for money to be essential. To preclude bilateral credit it is sufficient that a given pair of agents will meet again with probability zero. This is achieved in the current model by a random

[24]U.S. sales by interactive direct marketing (in person or by phone) are only 8% of GDP (Direct Marketing Association, 2002). Most of that is undoubtedly attributed to the existence of homebound homemakers and retirees, who are not modeled here. Also see footnote 14. A rare case of door-to-door sales being exceedingly popular is that of Russia in the early 1990s. The shortages in shops, typical of the Communist era, persisted for a while. Trade liberalization allowed people for the first time to import goods and sell them door-to-door to make up for these shortages. When the shortages were gone, so did the popularity of door-to-door sales. I thank Alexei Deviatov for this information.

choice of a particular shop among the many shops that offer the desired good. Any more randomness than this, while mathematically convenient, can come at a cost in terms of the quality and quantity of results[25]. At the other extreme, a well-organized system of deterministic trading posts is also not necessary. This semi-random model may be thought of as a missing link between random matching models and trading post models.

As in random matching models, a commodity or fiat object can endogenously function as money, but the details are more consistent with monetary history. First, any good can be a unique commodity money. Second, the good with the best intrinsic properties is the most likely to become commodity money. Third, fiat money can totally crowd out commodity money, even when some goods are more fit to be commodity money than others. These results hold for both production structures of Kiyotaki and Wright (1989).

Other interesting results are obtained by the richer structure of this model: agents might coordinate on Walrasian, money-less equilibria; the best commodity money equilibrium can be more likely to exist than any Walrasian equilibrium; agents are more likely to carry commodity/fiat money to shops where their consumption goods are held ("go shopping"), than to wait at home with the money for door-to-door salespersons; fiat money never Pareto dominates commodity money and Walrasian equilibria.

The model is kept as close as possible to Kiyotaki and Wright (1989) to show in the clearest possible way the differences between random matching and partially directed search. Future work should incorporate extensions as those made to random matching models, such as endogenous prices, divisible money, mixed strategies and direct barter. It is also possible to combine partially directed search and random matching in the same model by assuming that a searcher has a positive probability of randomly meeting other searchers between shops.

A benefit of the approach taken here is that all equilibria have much more order and discipline than in random matching models. It allows generalizing most of the results to the case of more than three types of goods and agents, even when goods have different intrinsic properties. More important are the implications for using the model to analyze broader macroeconomic issues. For example, combining this model with an assumption that the decision-making unit is a two-agent household, as suggested by Wallace (2002), creates a degenerate distribution of money in steady state. Such a model can be used to analyze long-run monetary policy (Goldberg [2006a]) and the propagation of real sectoral shocks (Goldberg [2006b]). The choice of a particular location can also be extended to include a choice of deliberately going to the bank, the bonds market (in anticipation of open market operations), the foreign exchange market, or the tax authority.

[25] Wallace (2002) defends the random matching assumption as one possible source of idiosyncratic uncertainty about consumption and earning opportunities. This uncertainty produces heterogeneity of money holdings, which is important for some issues in monetary economics.

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Appendix

A remark about mapping between propositions in this working paper and the abridged paper (forthcoming, Journal of Monetary Economics): To facilitate the presentation of the main issues and mechanisms, the JME paper starts with the case $c_1 = c_2 = c_3$ and then reports what changes if $c_1 < c_2 < c_3$. In the latter case there also arises the difference between Kiyotaki and Wright's Model A and Model B. This working paper analyzes all of these cases at once. The following table shows for each proposition in the JME paper where it is proved in the working paper.

JME paper	Working paper	JME paper	Working paper
1	1	14(i)	5
2	2	14(ii)	6
3	3	14(iii)	7
4	4	14(iv)	6, 7
5	5(i)	15	8
6	6(i), 7(i)	16	9
7	10	17	11
8	12	18	13
9	14	19(i)	6
10	15	19(ii)	7
11	16	19(iii)	8(ii)
12	1	19(iv)	13(ii)
13	4	20	14(ii)
		21	17

Proof of Lemma 1. **(i)** Searches and trades are voluntary. An agent can always choose to stay home forever and never produce or travel. With no costs, the value function equals zero at least. **(ii)** An agent who obtains his consumption good consumes it immediately due to discounting. If he obtains it in another city, immediate consumption also saves on the cost of transporting the good back home. An agent who can trade his production good for his consumption good does so because this gives an immediate gain of u without affecting future utility. As for an agent who can trade another good (not his production good) for his consumption good, consider the alternatives. 1. If he plans to hold that other good forever, his value function is non-positive. Trading now and consuming gives utility now, and the value function from then on is non-negative (he can stay home forever). 2. If he wants to trade that good for his production good later on, he will incur nothing but costs until that happens. Trading now and consuming enables him to produce his production good immediately and gain a net utility u. 3. If he plans to trade that good for his consumption good in the future, it is better to do it now due to discounting and possibly transport costs. $\square$

Proof of Proposition 1. Suppose that it is optimal for the agent to open a shop regardless of the good he holds. Then (1) and (2) reduce to $V_{ij} = \beta V_{ij}$ and $V_{ik} = \beta V_{ik}$, so $V_{ij} = V_{ik} = 0$. To verify the guess plug $V_{ij} = V_{ik} = 0$ in the right-hand-sides of (1) and (2) to get

$V_{ij} = \beta \max[0,\, -2c_j + \max(0, 0),\, -2c_j + 0] = 0,$

$V_{ik} = \beta \max[0,\, -2c_k + 0,\, -c_k + u + 0] = \beta \max[0,\, -c_k + u].$

Therefore, $V_{ij} = 0$ and staying home is optimal iff $u \leq c_k$. Since $V_{ij} = V_{ik}$ the agent indeed rejects goods that he does not consume, like everyone else. In conclusion, an agent's stay at home depends on the transport cost of the good that he neither consumes nor produces. Autarky is an equilibrium iff u is weakly smaller than any transport cost. In other words, $u \leq c_1$. $\square$

Proof of Proposition 2. Suppose that an agent believes that all other agents are searchers. The value functions are

$V_{ij} = \beta \max[p_1 V_{ij} + p_2(u + V_{ij}) + p_3 \max(V_{ij},\, V_{ik}),\, -2c_j + V_{ij},\, -2c_j + V_{ij}]$

$V_{ik} = \beta \max[p_4 V_{ik} + p_5(u + V_{ij}) + p_6 \max(V_{ik},\, V_{ij}),\, -2c_k + V_{ik},\, -2c_k + V_{ik}].$

The probabilities p_i refer to the matching possibilities of an agent who stays home. An agent who holds good j is not visited, or not visited by anyone who wants to trade, with probability p_1. He is visited by a searcher who wants to trade and holds good i with probability p_2, and he is visited by a searcher who wants to trade and holds good k with probability p_3. Searching is futile because the other cities are abandoned. Obviously it is better to stay home. $\square$

Proof of Proposition 3. See text.

Proof of Proposition 4. Consider the case where City j is the "market-place" where all the agents meet. Suppose that a type i agent believes that all other agents will meet there with their production goods and accept only their consumption goods. His optimal strategies are defined by the following equations.

$V_{ij} = \beta \max[V_{ij}, -c_j + u + V_{ij}, -2c_j + V_{ij}],$

$V_{ik} = \beta \max[V_{ik}, -2c_j + V_{ik}, -2c_k + V_{ik}].$

Staying home or going to City k will not result in trade because nobody is there. Going to City j with good j leads to participation in a trilateral Walrasian arrangement and consumption. Visiting City j with good k results in no trade. This is because by assumption trade must involve the stayer, and the stayer already has good k. Then $V_{ij} = \frac{\beta(u-c_j)}{1-\beta}$ and $V_{ik} = 0$. It is indeed optimal to search when holding good j iff $u > c_j$.

Consider now a type j agent. Given that he expects other types to visit his city with their production goods, his equations are[26]

$V_{jk} = \beta \max[-2c_k + V_{jk}, u + V_{jk}, -2c_k + V_{jk}]$,

$V_{ji} = \beta \max[-2c_i + V_{ji}, u + V_{jk}, -2c_i + V_{ji}]$.

An agent holding good k can get good j in the trilateral exchange if he stays, while if he travels he will find the other cities abandoned. If he holds good i and stays home he can trade it directly with the type i searcher and get good j (in which case the type k searcher gets nothing). Clearly, $V_{jk} = V_{ji} = \frac{\beta u}{1-\beta}$.

Type k's equations are

$V_{ki} = \beta \max[-2c_i + V_{ki}, -c_i + u + V_{ki}, V_{ki}]$,

$V_{kj} = \beta \max[-2c_j + V_{kj}, -c_j + u + V_{kj}, V_{kj}]$.

If he goes to City j with good i he will participate in the trilateral exchange and consume. If he goes there with good j he will barter directly with the type j stayer and consume. There is also a type i searcher there with good j, but he is assumed to reject the stayer's good (good k). If he goes to City j in either case, then $V_{ki} = \frac{\beta(u-c_i)}{1-\beta}$ and $V_{kj} = \frac{\beta(u-c_j)}{1-\beta}$. This is indeed optimal iff $u > c_i$ and $u > c_j$.

In conclusion, u needs to be larger than both c_j and c_i. $\square$

Proof of Proposition 5. All type k agents take good i to City j. Type j stayers who receive good i from them take good i to City i where they exchange it for good j. Only type j agents accept goods that they do not consume. As for off-equilibrium location strategies, type i agents stay home even if they hold good k, and type k agents go to City j even if they hold good j.

It is useful to start with a type j agent, an intermediary, who believes everyone follows these strategies. His equations are

$$V_{jk} = \beta \max[-2c_k + V_{jk}, \max(V_{jk}, V_{ji}), -2c_k + V_{jk}], \tag{A.1}$$

$$V_{ji} = \beta \max[-c_i + u + V_{jk}, V_{ji}, -2c_i + V_{ji}]. \tag{A.2}$$

If he holds good k he cannot trade in other cities because City k is abandoned and in City i everyone rejects good k. Therefore he opens a shop when holding good k. Since all type k agents visit City j, every type j stayer will be visited by one of them. He can trade and get good i if he wants to. If he takes good i to City i he can trade it for sure because all type i agents open shops and he is the only searcher in the shop he enters.

If he follows the equilibrium strategies (accepting good i at his shop and taking it to City i), the equations reduce to $V_{jk} = \beta V_{ji}$ and $V_{ji} = \beta[-c_i + u + V_{jk}]$. The solution is $V_{ji} = \frac{\beta(u-c_i)}{1-\beta^2}$ and $V_{jk} = \frac{\beta^2(u-c_i)}{1-\beta^2}$. Accepting good i in his shop

[26] Recall that in the squared brackets of the value functions, the order is always [go to City i, go to City j, go to City k]. This is the notation not only for type i, but for all types.

($V_{ji} > V_{jk}$) is optimal iff $u > c_i$. Taking it to City i is better than staying home ($-c_i + u + V_{jk} > V_{ji}$) iff $u > c_i$.

Note that all type j agents trade every period as they are on the short side of the market in both City j and City i. A fraction p_{jk} of them trade good k for good i; a fraction p_{ji} trade good i for good j, consume and produce a new unit of good k. In steady-state $p_{jk} = p_{ji} = .5$.

Type i's equations are

$$V_{ij} = \beta \max[p_{ji}(u + V_{ij}) + p_{jk}V_{ij},\, -c_j + p_{jk}(u + V_{ij}) + p_{ji}(V_{ij} - c_j),\\ -2c_j + V_{ij}], \tag{A.3}$$

$$V_{ik} = \beta \max[V_{ik},\, -2c_k + V_{ik},\, -2c_k + V_{ik}]. \tag{A.4}$$

If he opens a shop when holding good j, he is matched with a type j searcher (and trades) with probability p_{ji}. If he goes to City j, he visits an open shop with probability p_{jk}. Because there is also a type k searcher there, there is a trilateral Walrasian exchange. With probability p_{ji} he visits a factory of an agent who holds good i, but that agent is in City i that period. Going to City k is useless since it is abandoned. If he holds good k, staying home cannot lead to trade because type j searchers reject their own production good. If he goes to City j, the only stayers he may meet already hold good k. Obviously, $V_{ik} = 0$ so he stays home as conjectured.

With $p_{jk} = p_{ji} = .5$, equation (A.3) becomes

$$V_{ij} = \beta \max[.5u + V_{ij},\, -1.5c_j + .5u + V_{ij}],$$

so staying home is optimal. Note that $V_{ij} = \frac{.5\beta u}{1-\beta} > V_{ik}$, so type i indeed rejects good k as conjectured.

Type k's equations are

$$V_{ki} = \beta \max[-c_i + \max(V_{ki} - c_i,\, V_{kj} - c_j),\\ -c_i + p_{jk}(u + V_{ki}) + p_{ji}(V_{ki} - c_i),\, V_{ki}], \tag{A.5}$$

$$V_{kj} = \beta \max[-2c_j + V_{kj},\, -c_j + p_{jk}(u + V_{ki}) + p_{ji}\left(V_{kj} - c_j\right),\, V_{kj}]. \tag{A.6}$$

In City i there are more stayers than type j searchers, so if he goes there he is the only searcher in the shop he visits. Therefore he can surely get good j if he wants to. In City j he finds an open shop with probability p_{jk}. If he indeed goes to City j when holding good i then $V_{ki} = \frac{\beta(u-3c_i)}{2(1-\beta)}$, and it is better than staying home iff $u > 3c_i$. It also needs to be shown that going to City j is better than going to City i, and this is not trivial if $V_{kj} - c_j > V_{ki} - c_i$ (see equation (A.5)). Therefore, it is necessary to analyze the off-equilibrium location strategy of a type k agent holding good j.

Case 1: a holder of good j goes to City j. Then $V_{kj} = \beta\frac{(2-\beta)u_k - 3\beta c_i - 6(1-\beta)c_j}{2(1-\beta)(2-\beta)}$. It is optimal iff $u > \frac{3\beta c_i + 6(1-\beta)c_j}{2-\beta} \equiv X$. Then a holder of good i indeed goes to City j rather than City i iff $u > \frac{(2+5\beta)c_i - 4(1+\beta)c_j}{2-\beta} \equiv Y$.

Case 2: a holder of good j stays home. Then $V_{kj} = 0$. It is optimal iff $u < X$. Then a holder of good i indeed goes to City j rather than City i iff $u > 3c_i \equiv Z$. Note that $u > Z$ is already a necessary condition above.

The following relations between these thresholds can be shown:

$Z > X$ iff $c_i > c_j$;

$X > Z > Y$ if $c_i < c_j$;

$X = Z > Y$ if $c_i = c_j$.

To sum up, if $c_i > c_j$ there is an additional necessary condition $u > Y$. $\square$

Proof of Proposition 6. Type j agents go first to City k to get good i. Then they are visited by type i searchers. Only type j agents accept goods that they do not consume. Type i's equations are

$V_{ij} = \beta \max[V_{ij},\, -c_j + p_{ji}(u + V_{ij}) + p_{jk}(V_{ij} - c_j),\, -2c_j + V_{ij}]$,

$V_{ik} = \beta \max[V_{ik},\, -2c_k + V_{ik},\, -c_k + u + V_{ij}]$.

If he goes to City k, where there are only p_{jk} other searchers, he is the only searcher in the shop he enters. Going to City j when holding good j yields $V_{ij} = \frac{\beta(u-3c_j)}{2(1-\beta)}$. It is better than staying home iff $u > 3c_j \equiv F$. Type j agents bother going to City k and get good i only if they believe that type i agents reject good k if it is offered to them in City j (i.e., $V_{ij} - c_j \geq V_{ik} - c_k$). There are two possible cases regarding the off-equilibrium location strategy (V_{ik}).

Case 1: a holder of good k stays home. Then $V_{ik} = 0$. It is optimal iff it is weakly better than going to City k, or $0 \geq -c_k + u + V_{ij}$. This holds iff $u \leq \frac{3\beta c_j + 2(1-\beta)c_k}{2-\beta} \equiv G$. Then $V_{ij} - c_j \geq V_{ik} - c_k$ iff $u \geq \frac{(2+\beta)c_j - 2(1-\beta)c_k}{\beta} \equiv H$.

Case 2: a holder of good k goes to City k. Then $V_{ik} = \beta \frac{(2-\beta)u - 3\beta c_j - 2(1-\beta)c_k}{2(1-\beta)}$ and it is indeed optimal iff $u > G$. Then $V_{ij} - c_j \geq V_{ik} - c_k$ iff $u \leq \frac{2(1-\beta)c_k - (2+3\beta)c_j}{\beta} \equiv I$.

Type j's equations are

$V_{jk} = \beta[-2c_k + V_{jk},\, V_{jk},\, -c_k + \max(V_{jk} - c_k, V_{ji} - c_i)]$,

$V_{ji} = \beta[-2c_i + V_{ji},\, u_j + V_{jk},\, -2c_i + V_{ji}]$.

Equilibrium strategies imply $V_{jk} = \beta \frac{\beta u - (c_i + c_k)}{1-\beta^2}$ and $V_{ji} = \beta \frac{u - \beta(c_i + c_k)}{1-\beta^2}$. Searching when holding good k is better than staying home iff $u > \frac{c_i + c_k}{\beta} \equiv J$. Trading good k for good i in City k ($V_{ji} - c_i > V_{jk} - c_k$) is optimal iff $\beta u > c_i - (1+2\beta)c_k$. This holds if $u > J$.

Type k's equations are

$V_{ki} = \beta[-2c_i + V_{ki},\, -2c_i + V_{ki},\, p_{jk}(u + V_{ki}) + p_{ji}V_{ki}]$,

$V_{kj} = \beta[-2c_j + V_{kj},\, -c_j + p_{ji}\max(V_{kj} - c_j, V_{ki} - c_i) + p_{jk}(V_{kj} - c_j),$
$\quad p_{jk}(u + V_{ki}) + p_{ji}V_{kj}]$.

Clearly, staying home is optimal when holding good k, so $V_{ki} = \frac{\beta u}{2(1-\beta)}$. Staying home is also optimal when holding good j. To see this note that it may be optimal to go to City j only if he wants to trade good j for good i ($V_{ki} - c_i > V_{kj} - c_j$). Then staying home is optimal iff

$-c_j + p_{ji}(V_{ki} - c_i) + p_{jk}(V_{kj} - c_j) \leq p_{jk}(u + V_{ki}) + p_{ji}V_{kj}$.

As in the shopping equilibrium, $p_{ji} = p_{jk} = .5$, so the inequality becomes $-c_j - .5c_i - .5c_j \leq .5u$, which always holds. Note from type k's equations that if he stays home no matter what he holds, then $V_{ki} = V_{kj}$. Because it is assumed that indifferent agents do not trade, he does not trade good i for good j if offered to him by a type i searcher. This is a critical ingredient in the construction of the equilibrium. It prevents type i agents from going to City k.

Collecting the conditions derived above for all types, we have:

1. $u > F$.
2. $u > J$.

3. If $u \leq G$ then $u \geq H$.

4. If $u > G$ then $u \leq I$.

It can be shown that $F \geq G$ iff $3c_j \geq c_k$, and that if $3c_j < c_k$ then $J > G$. This means that Condition 3 is irrelevant. Condition 4 can hold only if $G < I$, which simplifies to $c_j < (1-\beta)c_k$. It cannot hold when $c_j \geq c_k$. To summarize, the equilibrium exists iff $\max(F, J) < u \leq I$. It can be shown that $J < I$ only if $\beta < .5$.

Returning to type k, recall that he rejects good j at home only because he is indifferent between accepting and rejecting. Under an alternative assumption of accepting good j when indifferent, type i's equation, when holding good j becomes

$V_{ij} = \beta \max[V_{ij},\, -c_j + p_{ji}(u + V_{ij}) + p_{jk}(V_{ij} - c_j),\, -c_j + u + V_{ij}]$,

It is easy to show that it is optimal to go to City k instead of City j, thus breaking the equilibrium. $\square$

Proof of Proposition 7. Types i and k open shops and reject goods that they do not consume. Type j agents go first to City k to get good i, return home, and then go to City i to get good j. Type i's equations are

$V_{ij} = \beta \max[p_{ji}(u + V_{ij}) + p_{jk}V_{ij},\, -2c_j + V_{ij},\, -2c_j + V_{ij}]$,

$V_{ik} = \beta \max[V_{ik},\, -2c_k + V_{ik},\, -c_k + u + V_{ij}]$.

Clearly, $V_{ij} = \frac{\beta u}{2(1-\beta)}$. Similarly to the door-to-door equilibrium, it must be verified that if a type j searcher offers good k to type i, type i will reject it ($V_{ij} \geq V_{ik}$). Otherwise, type j agents will go only to City i. There are two cases.

Case 1: a holder of good k stays home. Then $V_{ik} = 0 < V_{ij}$. It is optimal iff $u \leq \frac{2(1-\beta)c_k}{2-\beta}$.

Case 2: a holder of good k goes to City k. Then $V_{ik} = \beta\frac{(2-\beta)u - 2(1-\beta)c}{2(1-\beta)}$. It is optimal iff $u > \frac{2(1-\beta)c_k}{2-\beta}$. Then $V_{ij} \geq V_{ik}$ iff $u \leq 2c_k$.

Since $\frac{2(1-\beta)c_k}{2-\beta} < 2c_k$, the necessary condition that summarizes both cases is $u \leq 2c_k$.

Type j's equations are

$V_{jk} = \beta \max[-2c_k + V_{jk},\, V_{jk},\, -c_k + \max(V_{jk} - c_k,\, V_{ji} - c_i)]$,

$V_{ji} = \beta \max[-c_i + u + V_{jk},\, V_{ji},\, -2c_i + V_{ji}]$.

Equilibrium strategies yield $V_{jk} = \beta\frac{\beta u - (1+\beta)c_i - c_k}{1-\beta^2}$ and $V_{ji} = \beta\frac{u - (1+\beta)c_i - \beta c_k}{1-\beta^2}$. The conditions are:

1. Exchanging good k for good i ($V_{ji} - c_i > V_{jk} - c_k$) is optimal iff $\beta u > (1+\beta)c_i - c_k$.

2. Going to City k when holding good k is optimal iff $\beta u > (1+\beta)c_i + c_k$.

3. Going to City i when holding good i is optimal iff $u > (1+\beta)c_i + \beta c_k$.

If Condition 2 holds, the others hold as well.

Type k's equations are

$V_{ki} = \beta[-c_i + \max(V_{ki} - c_i,\, V_{kj} - c_j),\, -2c_i + V_{ki},\, p_{jk}(u + V_{ki}) + p_{ji}V_{ki}]$,

$V_{kj} = \beta[-2c_j + V_{kj},\, -2c_j + V_{kj},\, p_{jk}(u + V_{ki}) + p_{ji}V_{kj}]$.

Always staying home yields $V_{ki} = V_{kj} = \frac{\beta u}{2(1-\beta)}$. As in the door-to-door equilibrium, this indifference leads to rejection of good j. It is easy to verify that always staying home is indeed optimal.

To sum up, the necessary and sufficient conditions for the existence of the one-searcher equilibrium are:

1. $u \leq 2c_k$.
2. $u > \frac{(1+\beta)c_i + c_k}{\beta}$.

These conditions are consistent with each other iff $\frac{(1+\beta)c_i+c_k}{\beta} < 2c_k$, or $(1+\beta)c_i < (2\beta - 1)c_k$. This holds only if $c_i < c_k$.

Going back to type k's indifference, if it is assumed that indifference leads to trade, then type i's equation when holding good j becomes

$V_{ij} = \beta \max[p_{ji}(u + V_{ij}) + p_{jk}V_{ij}, -2c_j + V_{ij}, -c_j + u + V_{ij}]$.

Staying home is still optimal iff $u \leq 2c_j$. $\square$

Proof of Proposition 8. (i) Let *Shopping(i)* denote the shopping equilibrium with good i as money, and similarly for the other types of commodity money equilibria.

Step 1: consider Model A. Shopping(1) exists iff $u > 3c_1$. Door-to-door(2) does not exist (Proposition 6). One-searcher(2) and One-searcher(3) do not exist (Proposition 7). Propositions 6 and 7 also imply that a necessary condition for the existence of any other equilibrium, except One-searcher(1), is $u > 3c_1$, or $u > 3c_2$, or $u > 3c_3$. Therefore, if they exist, Shopping(1) exists too. All that is left to prove for Model A is that if One-searcher(1) exists then Shopping(1) exists as well.

Step 2: One-seracher(1) exists iff $\frac{(1+\beta)c_1+c_3}{\beta} < u \leq 2c_3$. This requires $\frac{(1+\beta)c_1+c_3}{\beta} < 2c_3$, which reduces to $c_1 < \frac{(2\beta-1)c_3}{1+\beta}$. Since $c_1 \geq 0$, One-searcher(1) exists only if $\beta > 1/2$. Suppose that indeed $\beta > 1/2$. Also suppose that $u = 3c_1 + \varepsilon$, and plug it in $\frac{(1+\beta)c_1+c_3}{\beta} < u \leq 2c_3$. The left inequality reduces to $\frac{c_3-\beta\varepsilon}{2\beta-1} < c_1$ and the right inequality becomes $c_1 \leq \frac{2c_3-\varepsilon}{3}$. These last two conditions are consistent with each other iff $\frac{c_3-\beta\varepsilon}{2\beta-1} < \frac{2c_3-\varepsilon}{3}$, which reduces to $\varepsilon > \frac{(5-4\beta)c_3}{1+\beta}$. Note that the right hand side is strictly positive, so One-searcher(1) exists only if ε is larger enough than zero. However, Shopping(1) exists for all $\varepsilon > 0$. To sum up, $u > 3c_1$ is a necessary condition for the existence of all equilibria, but it is also a sufficient condition only for Shopping(1).

Step 3: for Model B the analysis is similar. Again, a problem may arise only in the one-searcher equilibria. As for One-searcher(1), simply replace c_3 by c_2 in Step 2. As for One-searcher(2), it requires $u > \frac{(1+\beta)c_2+c_3}{\beta}$, and the right hand side is trivially larger than $3c_1$. One-searcher(3) does not exist (Proposition 7).

(ii) Shopping(3) exists only if $u > 3c_3$. If this holds, then obviously $u > 3c_2$ (this is the only condition for Shopping(2)) and $u > 3c_1$ (this is the only condition for Shopping(1)).

Door-to-door(3) exists only if $u > \frac{c_3+c_2}{\beta}$ and $u > 3c_1$. These conditions do *not* necessarily imply that $u > 3c_2$ (a necessary condition for Door-to-door(1)).

As for Model B, Shopping(3) exists only if $u > 3c_3$. This does *not* necessarily imply that $u > \frac{(2+5\beta)c_2-4(1+\beta)c_1}{2-\beta}$ (a necessary condition for Shopping(2)). One-searcher(2) exists only if $u \leq 2c_3$. This does *not* necessarily imply that $u \leq 2c_2$

(a necessary condition for One-searcher(1). $\square$

Proof of Proposition 9. Let expected welfare for a type i agent be W_i^c in a commodity money equilibrium. Let the discount rate be $r \equiv 1/\beta - 1$. Table A.1 shows individual and aggregate welfare for all commodity money equilibria. The column "$r\Sigma W^c$" proves the proposition. $\square$

Table A.1: Welfare with Good i as Commodity Money

Equilibrium	Searchers [carry]	rW_i^c	rW_j^c	rW_k^c	$r\Sigma W^c$
Shopping	$k[i]$, $j[i]$	$\frac{u}{2}$	$\frac{u-c_i}{2}$	$\frac{u-3c_i}{2}$	$\frac{3u-4c_i}{2}$
Door-to-door	$j[k]$, $i[j]$	$\frac{u-3c_j}{2}$	$\frac{u-c_i-c_k}{2}$	$\frac{u}{2}$	$\frac{3u-(c_i+3c_j+c_k)}{2}$
One-searcher	$j[k, i]$	$\frac{u}{2}$	$\frac{u-2c_i-c_k}{2}$	$\frac{u}{2}$	$\frac{3u-(2c_i+c_k)}{2}$

Proof of Proposition 10. **(i)** With symmetric transport costs, all the Walrasian equilibria exist iff $u > c$ (Proposition 4). All the shopping equilibria exist iff $u > 3c$ (Proposition 5). Then if the latter exist, the Walrasian equilibria exist as well. All door-to-door equilibria and one-searcher equilibria do not exist (Propositions 6, 7).

(ii) Let expected welfare for a type i agent be W_i^w in a Walrasian equilibrium. In the Walrasian equilibrium in which all trade is in City i, $rW_i^w = u$, and $rW_j^w = rW_k^w = u - c$. Compare this to Table A.1 (row "Shopping") to see the Pareto dominance.

(iii) In any Walrasian equilibrium aggregate welfare is $r\Sigma W^w = 3u - 2c$, while in any shopping equilibrium aggregate welfare is $r\Sigma W^c = 1.5u - 2c$. $\square$

Proof of Proposition 11. Consider Model A. **(i)** Shopping(1) exists iff $u > 3c_1$ (Proposition 5). According to Proposition 4, the Walrasian equilibrium in City 1 or 3 exists iff $u > c_3$, and the Walrasian equilibrium in City 2 exists iff $u > c_2$. If $c_1 \leq c_2/3$, then Shopping(1) exists but no Walrasian equilibrium exists.

(ii) From Table A.1, in shopping(1), $rW_1^c = u/2$ and $rW_2^c = (u - c_1)/2$. In the Walrasian equilibrium in City 1 or 3, $rW_2^w = u - c_3$. Then $rW_2^w < rW_2^c$ iff $c_1 < 2c_3 - u$. In the Walrasian equilibrium in City 2, $rW_1^w = u - c_2$. Then $rW_1^w < rW_1^c$ iff $u < 2c_2$.

(iii) From Table A.1, in shopping(1), $r\Sigma W^c = \frac{3u-4c_1}{2}$. In the Walrasian equilibrium centered in City 3, aggregate welfare is $r\Sigma W^w = 3u - c_2 - c_3$. Then $r\Sigma W^w < r\Sigma W^c$ iff $3u < 2(c_2 + c_3 - 2c_1)$. This can be consistent with the necessary condition for this equilibrium's existence ($u > c_3$). $\square$

Proof of Proposition 12. Consider a hypothetical equilibrium in which both goods i and k are money. Good i can become money only if type j agents holding good k meet type k agents holding good i. Good k can become money only if type i agents holding good j meet type j agents holding good k. This means that type j agents holding good k need to meet both type k agents and type i agents such that they can trade bilaterally with either one of them. With pure, stationary location strategies, this can happen only if type j agents holding good k always stay home and are visited by the other types. But this is the Walrasian, money-less equilibrium of City j. $\square$

Proof of Proposition 13. Table A.2 describes in detail for each commodity money equilibrium that can exist in Model A, where trades are made and which goods are traded in each match.

Table A.2

Equilibrium	First trade is in City	Goods traded there	Second trade is in City	Goods traded there
Shopping(1)	2	1, 3	1	1, 2
Shopping(2)	3	1, 2	2	2, 3
Shopping(3)	1	2, 3	3	1, 3
Door-to-door(1)	3	1, 3	2	1, 2
Door-to-door(3)	2	2, 3	1	1, 3
One-searcher(1)	3	1, 3	1	1, 2

(i) Table A.3 reorganizes the data of Table A.2, and shows that for every pair of city and good, there is a commodity money equilibrium in which that good is traded in that city.

Table A.3

City	Good	Equilibrium
1	1	Shopping(1), Door-to-Door(3), One-searcher(1)
1	2	Shopping(1), Shopping(3), One-searcher(1)
1	3	Shopping(3), Door-to-Door(3)
2	1	Shopping(1), Door-to-Door(1)
2	2	Shopping(2), Door-to-Door(1), Door-to-Door(3)
2	3	Shopping(1), Shopping(2), Door-to-Door(3)
3	1	Shopping(2), Shopping(3), Door-to-Door(1), One-searcher(1)
3	2	Shopping(2)
3	3	Shopping(3), Door-to-door(1), One-searcher(1)

(ii) Table A.4 reorganizes that same data to show that almost every possible bilateral exchange happens in every city in some equilibrium.

Table A.4

City	Goods exchanged	Equilibrium
1	1, 2	Shopping(1), One-searcher(1)
1	1, 3	Door-to-door(3)
1	2, 3	Shopping(3)
2	1, 2	Door-to-door(1)
2	1, 3	Shopping(1)
2	2, 3	Shopping(2), Door-to-door(3)
3	1, 2	Shopping(2)
3	1, 3	Shopping(3), Door-to-door(1), One-searcher(1)
3	2, 3	-

Tables A.5-A.7 do the same for Model B.

Table A.5

Equilibrium	First trade is in City	Goods traded there	Second trade is in City	Goods traded there
Shopping(1)	3	1, 2	1	1, 3
Shopping(2)	1	2, 3	2	1, 2
Shopping(3)	2	1, 3	3	2, 3
Door-to-door(2)	3	2, 3	1	1, 2
One-searcher(1)	2	1, 2	1	1, 3
One-searcher(2)	3	2, 3	2	1, 2

Table A.6

City	Good	Equilibrium
1	1	Shopping(1), Door-to-Door(2), One-searcher(1)
1	2	Shopping(2), Door-to-Door(2)
1	3	Shopping(1), Shopping(2), One-searcher(1)
2	1	Shopping(2), Shopping(3), One-searcher(1), One-searcher(2)
2	2	Shopping(2), One-searcher(1), One-searcher(2)
2	3	Shopping(3)
3	1	Shopping(1)
3	2	Shopping(1), Shopping(3), Door-to-door(2), One-searcher(2)
3	3	Shopping(3), Door-to-door(2), One-searcher(2)

Table A.7

City	Goods exchanged	Equilibrium
1	1, 2	Door-to-door(2)
1	1, 3	Shopping(1), One-searcher(1)
1	2, 3	Shopping(2)
2	1, 2	Shopping(2), One-searcher(1), One-searcher(2)
2	1, 3	Shopping(3)
2	2, 3	-
3	1, 2	Shopping(1)
3	1, 3	-
3	2, 3	Shopping(3), Door-to-door(2), One-searcher(2)

(iii) From Proposition 4, in every city there can be a Walrasian equilibrium.

(iv) From Proposition 1, there can be an autarky equilibrium. □

Proof of Proposition 14. If a type i agent follows the equilibrium strategies, then $V_{i0} = [m\beta^2 + \beta(1-\beta)]\frac{(1-m)u-(1+m)c_0}{1-\beta}$ and $V_{ij} = m\beta^2 \frac{(1-m)u-(1+m)c_0}{1-\beta}$. When he holds fiat money, going to City k is better than staying home iff $u > \frac{(1+m)c_0}{1-m} \equiv A$. This is also the condition for accepting fiat money at home ($V_{i0} > V_{ij}$). Other conditions depend on the off-equilibrium strategy as a holder of good k.

Case 1: a holder of good k goes to City k. Then

$V_{ik} = \beta\frac{(1-m)[1-\beta+(1-m)m\beta^2]u-(1-m)^2 m\beta^2 c_0-(1-\beta)(1+m)c_k}{(1-\beta)(1-m\beta)}$. It is better than staying home iff $u > \frac{(1-m^2)m\beta^2 c_0+(1-\beta)(1+m)c_k}{(1-m)[1-\beta+(1-m)m\beta^2]} \equiv B_i$. There are three conditions to verify about rejection of commodity money (good k): At home when holding good j ($V_{ij} \geq V_{ik}$) iff $u \leq \frac{(1+m)(c_k-m\beta c_0)}{(1-m)(1-m\beta)} \equiv C_i$; in City j when holding good j ($V_{ij} - c_j \geq V_{ik} - c_k$) iff $u \leq \frac{(1+\beta)c_k-(1-m\beta)c_j-m\beta^2(1+m)c_0}{\beta(1-m)(1-m\beta)} \equiv D_i$; and when holding fiat money at home ($V_{i0} \geq V_{ik}$) or in City j ($V_{i0} - c_0 \geq V_{ik} - c_k$) iff $c_0 \leq c_k$. The latter is apparent from comparing equations (3) and (5), if in both cases the agent goes to City k.

Case 2: a holder of good k stays home. Then $V_{ik} = 0$. It is optimal iff $u \leq B_i$. The other conditions: $V_{ij} \geq V_{ik}$ if $u > A$; $V_{ij} - c_j \geq V_{ik} - c_k$ iff $u \geq \frac{m\beta^2(1+m)c_0+(1-\beta)(c_j-c_k)}{m\beta^2(1-m)} \equiv E_i$; $V_{i0} - c_0 \geq V_{ik} - c_k$ if $u > A$ and $c_0 < c_k$.

Therefore, the necessary and sufficient conditions for the existence of the fiat equilibrium can be summarized as follows. First, $u > A$; second, if $u > B_i$, it needs to be the case that $u \leq \min(C_i, D_i)$ and $c_0 \leq c_k$. However, if $u \leq B_i$, then it needs to be the case that $u \geq E_i$.

(i) Figure 4 shows that for type 1 in Model A, all these equilibria conditions are consistent with each other. It assumes $\beta = 0.9$, $m = 0.5$, $c_n = 1 + .1n$ for $n = 1, 2, 3$. The combinations of (c_0, u) that satisfy all these conditions are in the triangle created by the vertical axis and the two bold lines.[27] The figures

[27] Note that the equilibrium can exist even if $u < c_n$ $\forall n$. In this case fiat money prevents autarky from being the unique equilibrium, because no commodity money equilibrium or Walrasian equilibrium exists.

for the other types in Model A, and for all types in Model B, are similar.

Because some of the equilibria in this paper do not exist with symmetric transport costs, it is useful to verify that the fiat shopping equilibrium does not have the same problem. It can be proven that $C_i = D_i$ and $E_i = A$ for all parameter values. Figure 5 shows what happens if in the example above $c_i = c_j = c_k = 1$.

(ii) In Case 1, one of the necessary conditions is $c_0 \leq c_k$. Case 2 holds iff $u \leq B_i$, which must be consistent with the general condition $u > A$. This requires $A < B_i$, which turns out to hold iff $c_0 \leq c_k$. Therefore, for a type i agent it must be the case that $c_0 \leq c_k$. This holds for all types iff $c_0 \leq c_1$. $\square$

Proof of Proposition 15. (i) In the shopping fiat equilibrium expected welfare for any type i, W_i^f, satisfies $rW_i^f = m(1-m)u - m(1+m)c_0$. The highest individual welfare in the commodity money equilibria satisfies $rW_i^c = .5u$. Since $m(1-m) < .5$, it is always true that $W_i^c > W_i^f$. In the Walrasian equilibria the highest individual welfare satisfies $rW_i^w = u$, so clearly $W_i^w > W_i^f$.

(ii) In the shopping fiat equilibrium aggregate welfare, ΣW^f, satisfies $r\Sigma W^f = 3m[(1-m)u - (1+m)c_0]$. The best case scenario for welfare in this equilibrium is when $c_0 = 0$ and $m = .5$. The latter minimizes matching frictions (i.e., maximizes $m(1-m)$). I will show that even in this case it is dominated by all commodity money equilibria.

$>$From Table A.1, it is better than a shopping commodity money equilibrium with good i as money iff $3m[(1-m)u - (1+m)c_0] > (3u - 4c_i)/2$, or $8c_i > 3u$. This contradicts a necessary condition for existence of the shopping commodity money equilibrium ($u > 3c_i$).

The shopping fiat equilibrium is better than a door-to-door equilibrium with good i as money iff $u < \frac{2}{3}(c_i + 3c_j + c_k) \equiv K$. Recall that the door-to-door equilibrium exists only if $J < u \leq I$ (see proof of Proposition 6). The shopping fiat equilibrium is better only if $J < K < I$. Now $J < I$ iff $c_j < \frac{(1-2\beta)c_k - c_i}{2+3\beta}$, while $J < K$ iff $c_j > \frac{(3-2\beta)(c_i+c_k)}{6\beta}$. These two conditions regarding c_j are consistent with each other iff $[6(1-\beta^2) + 11\beta]c_i < (-6 + \beta - 6\beta^2)c_k$, and this cannot hold.

The shopping fiat equilibrium is better than a one-searcher equilibrium with good i as money iff $4c_i + 2c_k > 3u$. This contradicts a necessary condition for the existence of the one-searcher equilibrium ($u > \frac{(1+\beta)c_i + c_k}{\beta}$).

Proof of Proposition 16. A type 1 agent's equations are

$$V_{i0} = \beta \max[(1-m)(u+V_{ij}) + mV_{i0},\, -2c_0 + V_{i0},\, -2c_0 + V_{i0}], \tag{A.7}$$

$$V_{ij} = \beta \max[V_{ij},\, -c_j + m \max(V_{ij} - c_j,\, V_{i0} - c_0) + (1-m)(V_{ij} - c_j),\, -2c_j + V_{ij}], \tag{A.8}$$

$$V_{ik} = \beta \max[(1-m)(u+V_{ij}) + mV_{ik},\, -2c_k + V_{ik},\, -c_k + m \max(V_{ik} - c_k,\, V_{i0} - c_0) + (1-m)(V_{ik} - c_k)]. \tag{A.9}$$

Clearly, a holder of fiat money stays home so (A.7) becomes

$$V_{i0} = \beta[(1-m)V_{ij} + mV_{i0} + (1-m)u], \tag{A.7'}$$

If he follows the equilibrium strategies, (A.8) becomes

$$V_{ij} = \beta[(1-m)V_{ij} + mV_{i0} - mc_0 - (2-m)c_j]. \tag{A.8'}$$

Comparing (A.7') and (A.8') shows that $V_{i0} > V_{ij}$.

Good k is rejected at home iff $V_{i0} \geq V_{ik}$, and in City j iff $V_{ij} - c_j \geq V_{ik} - c_k$. This requires solving V_{ik}, and there are two cases to consider.

Case 1. a holder of good k stays home. Then (A.9) becomes

$V_{ik} = \beta[(1-m)(u + V_{ij}) + mV_{ik}]$.

Compare with (A.7') to see that $V_{i0} = V_{ik}$. Then $V_{ij} - c_j \geq V_{ik} - c_k$ iff $c_k > \beta[mc_0 + (2-m)c_j + (1-m)u] + c_j$. This can be only if $c_k > c_j$, but such a ranking is impossible for all types.

Case 2. a holder of good k goes to City k. Because it requires $V_{ik} - c_k < V_{i0} - c_0$, (A.9) becomes

$V_{ik} = \beta[(1-m)V_{ik} + mV_{i0} - mc_0 - (2-m)c_k]$.

It is optimal, according to (A.9), iff

$(1-m)(u + V_{ij}) + mV_{ik} < (1-m)V_{ik} + mV_{i0} - mc_0 - (2-m)c_k$.

The right hand side is V_{ik}/β, so the inequality can be rewritten as $\frac{\beta(1-m)(u+V_{ij})}{1-m\beta} < V_{ik}$. By (A.7'), the left hand side of this latter inequality is V_{i0}, and this contradicts the necessary condition that $V_{i0} \geq V_{ik}$. $\square$

Proof of Proposition 17.

Generalizing the proofs for autarky and Walrasian equilibria is trivial. As for fiat money proofs, to ease on notation they are given below for Model A. Model B is similar.

Shopping equilibrium: Suppose that there are N types of agents. A type i agent consumes good i and produces good $i+1$ (mod N) for $i = 1, ..., N$. In all the equations below the first element in the brackets is the value of staying home. The equation of a fiat money holder is

$$\begin{aligned} V_{i0} = \beta[&V_{i0}, \\ &-c_0 + (1-m)\max(V_{i0} - c_0,\, V_{ij+1} - c_{j+1}) + m(V_{i0} - c_0), \\ &-c_0 + (1-m)\max(u_i + V_{ii+1}) + m(V_{i0} - c_0)], \end{aligned}$$

where $j \neq i,\, i-1,\, 0$. The third element is the value of going to City $i-1$. The second element is the value of going to any other city.

The equation for a holder of good $i+1$ is

$$\begin{aligned} V_{ii+1} = \beta[&m\max(V_{ii+1},\, V_{i0}) + (1-m)V_{ii+1}, \\ &-c_{i+1} + (1-m)\max(V_{ii+1} - c_{i+1},\, V_{ii+2} - c_{i+2}) + m(V_{ii+1} - c_{i+1}), \\ &-2c_{i+1} + V_{ii+1}]. \end{aligned}$$

The second element is the value of going to City $i+1$. The third element is the value of going to any other city.

The equation for a holder of good $j \neq i,\, i+1,\, i-1,\, 0$, is

$V_{ij} = \beta[V_{ij},\, -c_j + (1-m)\max(V_{ij} - c_j,\, V_{ij+1} - c_{j+1}) + m(V_{ij} - c_j),\, -2c_j + V_{ij}]$.

The second element is the value of going to City j. The third element is the value of going to any other city.

The equation for a holder of good $i-1$ is

$$\begin{aligned} V_{ii-1} = \beta[&V_{ii-1},\, -2c_{i-1} + V_{ii-1}, \\ &-c_{i-1} + (1-m)\max(u_i + V_{ii+1}) + m(V_{ii-1} - c_{i-1})]. \end{aligned}$$

The third element is the value of going to City $i-1$. The second element is the value of going to any city except City i or $i-1$.

If a type i agent follows equilibrium strategies then $V_{i0} = [m\beta^2 + \beta(1-\beta)]\frac{(1-m)u-(1+m)c_0}{1-\beta}$ and $V_{ii+1} = m\beta^2\frac{(1-m)u-(1+m)c_0}{1-\beta}$. I deal here only with the case where the off-equilibrium strategy of a holder of good $i-1$ is to stay home (this generalizes Case 2 in the proof of Proposition 14). Then $V_{ii-1}=0$. This implies that a holder of any good $j\neq i-1$, i, $i+1$, also stays home and $V_{ij}=0$. The reason is the following. An agent that holds such a commodity money $j\neq i-1$ can either stay home or trade it in City j for commodity money $j+1$. Then he can either stay home or trade that in City $j+1$ for commodity money $j+2$, etc. Even if he exploits any trade opportunity he ends up holding commodity money $i-1$, and by supposition then he stays home without consuming. Therefore, it is strictly better to stay home whenever holding any commodity money and avoid all the transport costs.

The conditions are:

1. Staying home when holding good $i-1$: iff $u\leq\frac{(1-m^2)m\beta^2c_0+(1-\beta)(1+m)c_{i-1}}{(1-m)[1-\beta+(1-m)m\beta^2]}\equiv B_i$ (this is a generalization of B_i from the proof of Proposition 14).

2. Going to City $i-1$ when holding fiat money: iff $u>A$.

3. Rejecting good $i+2$ when bringing good $i+1$ to City $i+1$ ($V_{ii+1}-c_{i+1}\geq V_{ii+2}-c_{i+2}$): iff $u\geq\frac{m\beta^2(1+m)c_0+(1-\beta)(c_{i+1}-c_{i+2})}{m\beta^2(1-m)}\equiv E_i$. This implies that staying home with good $i+1$ is optimal.

4. Rejecting any commodity money when holding the production good at home ($V_{ii+1}\geq V_{ij}$, $j\neq i$, $i+1$, 0): if $u>A$.

5. Rejecting good $i+1$ when holding fiat money ($V_{i0}>V_{ii+1}$) at home: if $u>A$.

Rejecting good $j\neq i, i+1$ when holding fiat money in another city ($V_{i0}-c_0\geq V_{ij}-c_j$): if $u>A$ and $c_0<c_j$. This implies that a holder of fiat money prefers going to City $i-1$ to going to any other city.

Rejecting own production good when holding fiat money in other cities ($V_{i0}-c_0\geq V_{ii+1}-c_{i+1}$): if $u_i>A$ and $c_0<c_{i+1}$.

Clearly, nothing here is substantially different from the proof of Proposition 14.[28]

Door-to-door equilibrium: The equation of a fiat money holder is:

$$V_{i0}=\beta[(1-m)(u+V_{ii+1})+mV_{i0},\ -2c_0+V_{i0}]. \tag{A.10}$$

The second element is the value of going anywhere else.

$$\begin{aligned}V_{ii+1}=\beta[&V_{ii+1}, \\ &-c_{i+1}+m\max(V_{ii+1}-c_{i+1},\ V_{i0}-c_0)+(1-m)(V_{ii+1}-c_{i+1}), \\ &-2c_{i+1}+V_{ii+1}].\end{aligned} \tag{A.11}$$

The second element is the value of going to City $i+1$. The third element is the value of going anywhere else.

$$\begin{aligned}V_{ij}=\beta[&V_{ij},\ -c_j+m\max(V_{ij}-c_j,\ V_{i0}-c_0)+(1-m)(V_{ij}-c_j), \\ &-2c_j+V_{ij}],\end{aligned} \tag{A.12}$$

for all $j\neq i-1$, i, $i+1$.

[28] The off-equilibrium strategy of trading one commodity money for another is irrelevant with three types of goods, but here it could happen off-equilibrium and it must be conjectured for the equilibrium. Suppose the conjecture is that commodity money is never traded for another. Then $V_{ij}-c_j>V_{ij+1}-c_{j+1}$ $\forall j\neq i, i+1, i-1, 0$. According to the equation describing V_{ij}, this implies $V_{ij}=0$. but then the inequality reduces to $c_j<c_{j+1}$. This cannot hold for all j.

The second element is the value of going to City j. The third element is the value of going anywhere else.

$$V_{ii-1} = \beta[(1-m)(u+V_{ii+1})+mV_{ii-1}, \tag{A.13}$$
$$-c_{i-1} + m\max(V_{ii-1} - c_{i-1}, V_{i0} - c_0) + (1-m)(V_{ii-1} - c_{i-1})],$$
$$-2c_{i-1} + V_{ii-1}].$$

The second element is the value of going to City $i-1$. The third element is the value of going anywhere else.

Trivially, a fiat money holder stays home. Following equilibrium strategies yields

$V_{i0} = \beta \frac{(1-m)[1-\beta+m\beta^2(1-m)]u-\beta(1-m)(1-m\beta)[mc_0+(2-m)c_{i+1}]}{(1-\beta)(1-m\beta)}$ and

$V_{ii+1} = \beta \frac{(1-m)m\beta u-(1-m\beta)[mc_0+(2-m)c_{i+1}]}{1-\beta}$. A holder of good $i+1$ goes to City $i+1$ iff $u > \frac{(1-m\beta)[mc_0+(2-m)c_{i+1}]}{(1-m)m\beta} \equiv L$. Other conditions depend on the off-equilibrium strategies of a holder of good $i+2$.

Case 1: a holder of good $i+2$ stays home. This holds iff

$u \leq (1-m\beta)\frac{m[1-\beta+m\beta^2(1-m)]c_0+m\beta^2(1-m)(2-m)c_{i+1}+(1-\beta)(2-m)c_{i+2}}{(1-m)m\beta[1-\beta+m\beta^2(1-m)]} \equiv M$. This can be an equilibrium only if there exists u such that $L < u \leq M$. It is easy to verify that $L < M$ iff $c_{i+1} < c_{i+2}$.

Case 2: a holder of good $i+2$ goes to City $i+2$. This holds only if $V_{i0} - c_0 > V_{ii+2} - c_{i+2}$. Then equation (A.12) becomes

$$V_{ii+2} = \beta[-c_{i+2} + m(V_{i0} - c_0) + (1-m)(V_{ii+2} - c_{i+2})]. \tag{A.12'}$$

A type i searcher in City $i+1$ expects to be paid in fiat money and reject the good $i+2$ produced there. That is, $V_{ii+1} - c_{i+1} \geq V_{ii+2} - c_{i+2}$. Comparing (A.12') and (A.11) reveals that this holds iff $c_{i+1} \leq c_{i+2}$.

In conclusion, neither case allows $c_{i+1} > c_{i+2}$, but this is the case for type $i = N-1$ agents in Model A. Their production good (good $i+1 = N$) is heavier than the good produced by their customers (good 1). Therefore they do not reject good $i+2$, thus inducing type N agents to deviate. $\square$

Table 1: Commodity Money Equilibria

i	j	k	Meaning
1	2	3	Good 1 is money in Model A
2	3	1	Good 2 is money in Model A
3	1	2	Good 3 is money in Model A
1	3	2	Good 1 is money in Model B
2	1	3	Good 2 is money in Model B
3	2	1	Good 3 is money in Model B

Figure 1: Commodity Money and Walrasian Equilibria

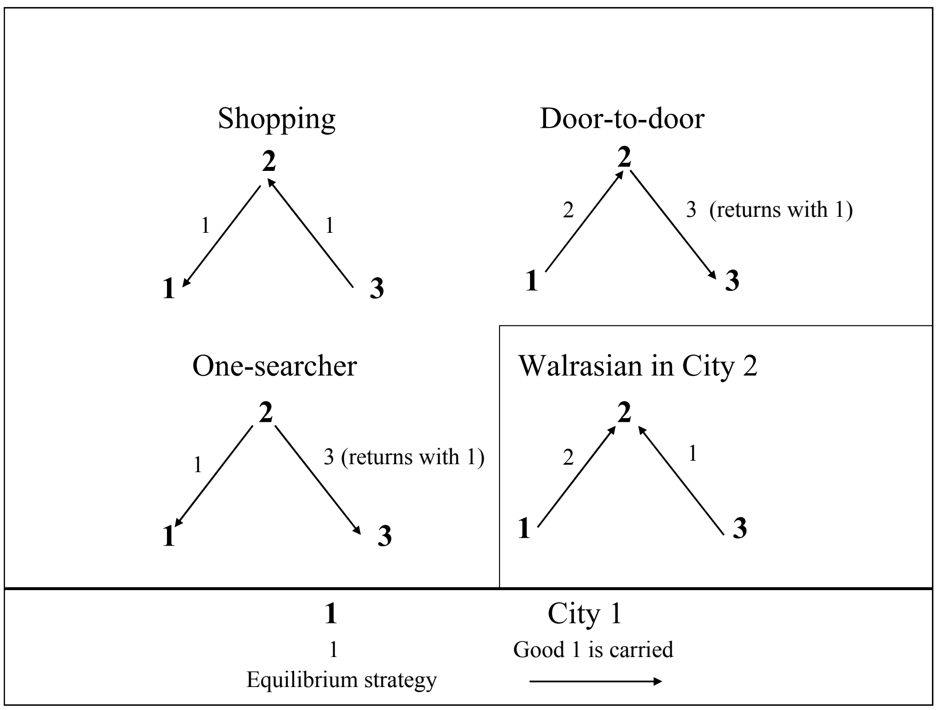


Figure 2: Deviation Strategies in Commodity Money Equilibria

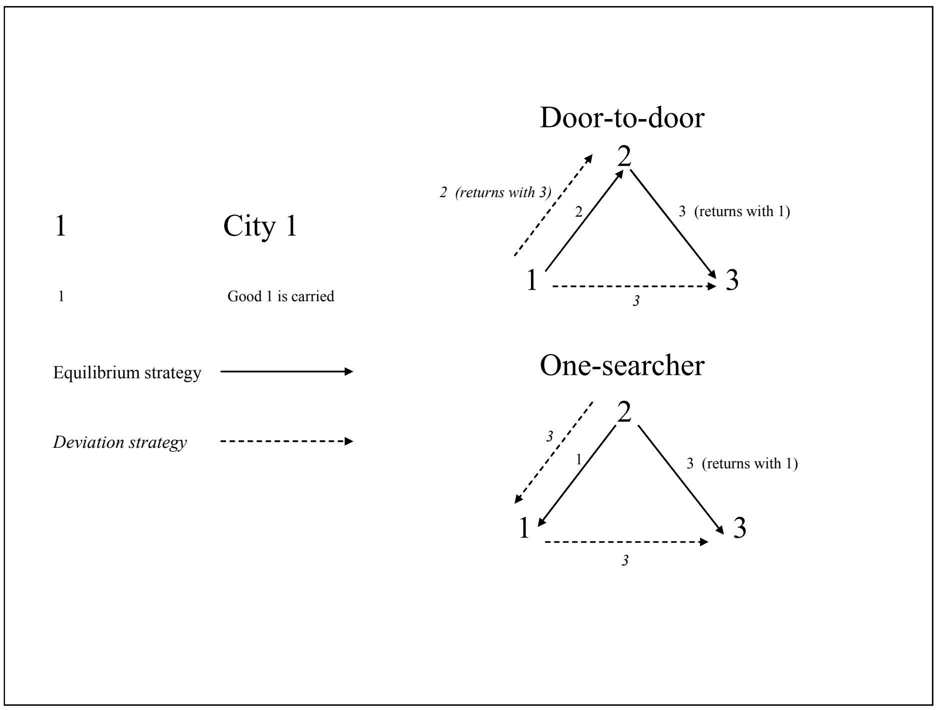
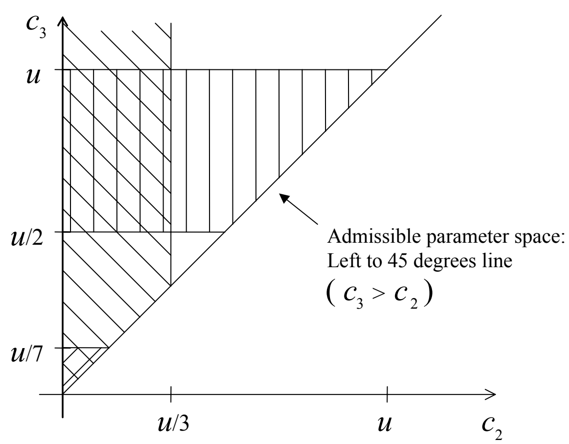


Figure 3: Shopping with the Best Commodity Money is the Most Likely Equilibrium



Necessary and sufficient equilibria conditions

Equilibrium \ Money	1	2	3
Shopping	All admissible space	$c_2 < u/3$	$c_3 < u/7$
One-searcher	$u/2 < c_3 < u$	Do not exist [Proposition 7(i)]	
Door-to-door	Do not exist [Proposition 6(iii)]		

Figure 4: Existence of the Shopping Fiat Equilibrium

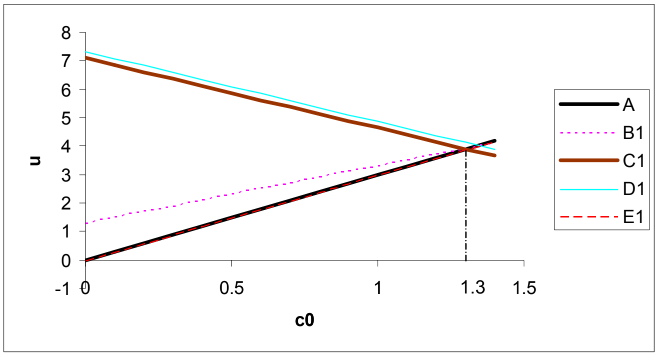


Figure 5: Shopping Fiat Equilibrium with Symmetric Transport Costs